Int J Theor Phys (2008) 47: 1354–1362
DOI 10.1007/s10773-007-9577-x

Noncommutativity, Generalized Uncertainty Principle and FRW Cosmology

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Received: 21 May 2007 / Accepted: 18 September 2007 / Published online: 10 October 2007

Abstract We consider the effects of noncommutativity and the generalized uncertainty principle on the FRW cosmology with a scalar field. We show that, the cosmological constant problem and removability of initial curvature singularity find natural solutions in this scenarios.

Keywords Noncommutativity · Generalised uncertainty principle · FRW cosmology

1 Introduction

Noncommutativity between spacetime coordinates which was first introduced in [1], has been attracting considerable attention in the recent past [2–6]. This renewed interest has its roots in the development of string and M-theories, [7, 8]. However, in all fairness, investigation of noncommutative theories may also be justified in its own right because of the interesting predictions regarding, for example, the IR/UV mixing and non-locality [9], Lorentz violation [10–12] and new physics at very short distance scales [13, 14]. The impact of noncommutativity in cosmology has also been considerable and addressed in different works [15–17]. Hopefully, noncommutative cosmology would lead us to the formulation of semiclassical approximations of quantum gravity and tackles the cosmological constant problem [18, 19]. Also it may solve the compactification of extra dimensions [19, 20]. To study the effects of noncommutativity in cosmology one can use two interesting different deformation of Poisson brackets, the Moyal product and the Generalized Uncertainty Principle (GUP), see for example [21–24]. The dynamical variable in general theory of relativity

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is the spacetime itself. Unfortunately, deformation of general relativity in noncommutative spacetime is a difficult task to analyze even simple models. However, to have a feeling of the effect of noncommutativity in cosmology, a proposal was made by considering directly a noncommutativity of minisuperspace of underling effective action. In this paper, as a simple toy model, we will assume noncommutativity among the gravitational and scalar fields in classical relativity using both of the above mentioned deformations. We will show that introducing noncommutativity with the Moyal product may solve the cosmological constant problem. On the other hand, GUP approach can remove the initial Big Bang curvature singularity.

2 Commutative Classical Cosmology

We consider a cosmological model in which the spacetime is assumed to be of FRW type. Thus the corresponding metric can be written as

$$ds^2 = -dt^2 + \frac{R^2(t)}{(1+\frac{k}{4}r^2)^2}(dr^2 + r^2 d\Omega^2), \tag{1}$$

where $k = 1, 0, -1$ denotes the usual spatial curvature and $R(t)$ is the scale factor of the universe.

Let us start from the Einstein-Hilbert action plus a scalar field as

$$\mathcal{S} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\mathcal{R} + \int d^4x\sqrt{-g}\left(-\frac{1}{2}(\nabla\phi)^2 - U(\phi)\right) + \mathcal{S}_{YGH}, \tag{2}$$

where $\kappa = 8\pi G$, $\mathcal{R}$ is Ricci scalar and $\mathcal{S}_{YGH}$ is the York-Gibbons-Hawking boundary term. The second term in (2) shows the usual scalar field action functional. Using the FRW line element (1) the action becomes

$$\mathcal{S} = \int dt\left[-3R\dot{R}^2 + 3kR + \left(\frac{1}{2}\dot{\phi}^2 - U(\phi)\right)R^3\right]. \tag{3}$$

In this paper, we will consider the case $k = 0$ and $U(\phi) = \Lambda$ such that the model describes a free scalar field with a cosmological constant Λ. The underlying mechanical analogue becomes more transparent if we make the following transformation [25]

$$\begin{cases} x_1 = R^{3/2}\cosh(\alpha\phi), \\ x_2 = R^{3/2}\sinh(\alpha\phi), \end{cases} \tag{4}$$

where $-\infty < \phi < \infty$, $0 \le R < \infty$ and $\alpha^2 = \frac{3}{8}$. Using the above transformations, the minisuperspace Lagrangian is given by

$$\mathcal{L} = \dot{x_1}^2 - \dot{x_2}^2 + 2\alpha^2\Lambda(x_1^2 - x_2^2). \tag{5}$$

Now, we can write the effective Hamiltonian as

$$\mathcal{H} = \frac{1}{4}(p_1^2 - p_2^2) - \omega^2(x_1^2 - x_2^2), \tag{6}$$

where p_i denotes the conjugate momenta corresponding to x_i and $\omega^2 = 2\alpha^2\Lambda$. Using the following commutative relation between x_i and p_i

$$\{x_i, p_j\} = \delta_{ij}, \quad i, j = 1, 2. \tag{7}$$

the canonical equations of motion become

$$\begin{cases} \dot{x}_1 = \frac{1}{2}p_1, & \dot{x}_2 = -\frac{1}{2}p_2, \\ \dot{p}_1 = 2\omega^2 x_1, & \dot{p}_2 = -2\omega^2 x_2, \end{cases} \tag{8}$$

where a dot denotes derivative with respect to t. Then we have the differential equations of motion

$$\ddot{x}_i - \omega^2 x_i = 0, \tag{9}$$

with solutions

$$x_i(t) = A_i e^{\omega t} + B_i e^{-\omega t}, \tag{10}$$

where A_i and B_i are integration constants. Note that the Hamiltonian constraint ($\mathcal{H} = 0$) imposes the following relation on these constants as

$$A_1 B_1 - A_2 B_2 = 0. \tag{11}$$

Finally, using (11) and choosing $A_1 = B_2$ and $A_2 = B_1$, the scale factor and scalar field are given by

$$\begin{cases} R(t)^3 = 2(A_1^2 - A_2^2)\sinh 2\omega t, \\ \phi(t) = \frac{1}{\alpha}\tanh^{-1}\left(\frac{A_2 e^{\omega t} + A_1 e^{-\omega t}}{A_1 e^{\omega t} + A_2 e^{-\omega t}}\right). \end{cases} \tag{12}$$

Note that if $\omega^2 = \frac{3}{4}\Lambda < 0$, the hyperbolic functions are replaced by their trigonometric counterparts in the above solutions.

3 Moyal Product Approach

We now concentrate on the noncommutativity concepts with Moyal product in phase space. The Moyal product in phase space may be traced to an early intuition by Wigner [26] which has been developing over the past decades [27–32]. Noncommutativity in classical physics [33, 34] is described by the Moyal product law between two arbitrary functions of positions and momenta as

$$(f \star_\alpha g)(x) = \exp\left[\frac{1}{2}\alpha^{ab}\partial_a^{(1)}\partial_b^{(2)}\right] f(x_1)g(x_2)\bigg|_{x_1=x_2=x}, \tag{13}$$

such that

$$\alpha_{ab} = \begin{pmatrix} \theta_{ij} & \delta_{ij} + \sigma_{ij} \\ -\delta_{ij} - \sigma_{ij} & \bar{\theta}_{ij} \end{pmatrix} \tag{14}$$

where the $N \times N$ matrices θ and $\bar{\theta}$ are assumed to be antisymmetric with $2N$ being the dimension of the classical phase space and σ can be written as a combination of θ and $\bar{\theta}$. With this product law, the deformed Poisson brackets can be written as

$$\{f, g\}_\alpha = f \star_\alpha g - g \star_\alpha f. \tag{15}$$

A simple calculation shows that

$$\{x_i, x_j\}_\alpha = \theta_{ij}, \qquad \{x_i, p_j\}_\alpha = \delta_{ij} + \sigma_{ij}, \qquad \{p_i, p_j\}_\alpha = \bar{\theta}_{ij}. \tag{16}$$

It is worth noting at this stage that, in addition to the noncommutativity in (x_1, x_2), we have also considered noncommutativity in the corresponding momenta. This should be interesting since its existence is due to essentially to the existence of noncommutativity in the space sector [27–29, 33, 34] and it would somehow be natural to include it in our considerations.

To proceed, we consider the following transformations on the classical phase space (x_i, p_j)

$$x_i' = x_i - \frac{1}{2}\theta_{ij} p^j, \qquad p_i' = p_i + \frac{1}{2}\bar{\theta}_{ij} x^j. \tag{17}$$

It is easy to check that if the (x_i, p_j) obey the usual Poisson algebra (7), then

$$\{x_i', x_j'\}_P = \theta_{ij}, \qquad \{x_i', p_j'\}_P = \delta_{ij} + \sigma_{ij}, \qquad \{p_i', p_j'\}_P = \bar{\theta}_{ij}. \tag{18}$$

These commutation relations are the same as (16). Consequently, to introduce noncommutativity, it is more convenient to work with Poisson brackets (18) than α-star deformed Poisson brackets (16). It is important to note that the relations represented by (16) are defined in the spirit of the Moyal product given above. However, in the relations defined in (18), the variables (x_i, p_j) obey the usual Poisson bracket relations so that the two sets of deformed and ordinary Poisson brackets represented by relations (16) and (18) should be considered as distinct.

Let us change the commutative Hamiltonian (6) with minimal variation to

$$\mathcal{H}' = \frac{1}{4}(p_1'^2 - p_2'^2) - \omega^2(x_1'^2 - x_2'^2), \tag{19}$$

where we have the commutation relations

$$\{x_i', x_j'\}_P = \theta\epsilon_{ij}, \qquad \{x_i', p_j'\}_P = (1+\sigma)\delta_{ij}, \qquad \{p_i', p_j'\}_P = \bar{\theta}\epsilon_{ij}, \tag{20}$$

with ϵ_{ij} being a totally anti-symmetric tensor and σ is given by

$$\sigma = \frac{1}{4}\bar{\theta}\theta. \tag{21}$$

We have also set $\theta_{ij} = \theta\epsilon_{ij}$ and $\bar{\theta}_{ij} = \bar{\theta}\epsilon_{ij}$. Using the transformation of (17), Hamiltonian (19) becomes

$$\mathcal{H} = \frac{1}{4}(1 + \omega^2\theta^2)(p_1^2 - p_2^2) - \left(\omega^2 + \frac{\bar{\theta}^2}{16}\right)(x_1^2 - x_2^2) + \left(\frac{\bar{\theta}}{4} + \theta\omega^2\right)(p_1 x_2 + p_2 x_1). \tag{22}$$

The equations of motion corresponding to the Hamiltonian (22) are given by

$$\begin{cases} \dot{x_1} = \{x_1, \mathcal{H}\}_P = \frac{1}{2}(1+\omega^2\theta^2)p_1 + \left(\frac{\bar{\theta}}{4} + \theta\omega^2\right)x_2, \\ \dot{x_2} = \{x_2, \mathcal{H}\}_P = \frac{-1}{2}(1+\omega^2\theta^2)p_2 + \left(\frac{\bar{\theta}}{4} + \theta\omega^2\right)x_1, \\ \dot{p_1} = \{p_1, \mathcal{H}\}_P = 2\left(\omega^2 + \frac{\bar{\theta}^2}{16}\right)x_1 - \left(\frac{\bar{\theta}}{4} + \theta\omega^2\right)p_2, \\ \dot{p_2} = \{p_2, \mathcal{H}\}_P = -2\left(\omega^2 + \frac{\bar{\theta}^2}{16}\right)x_2 - \left(\frac{\bar{\theta}}{4} + \theta\omega^2\right)p_1, \end{cases} \tag{23}$$

where we have used relations (7). Again It can be easily checked that if one writes the equations of motion for noncommutative variables, (20), with respect to the Hamiltonian (19) and uses transformation rules (17), one gets a linear combination of the equations of motion (23). This points to the fact that these two approaches are equivalent. Now, as a consequence of equations of motion (23), one has

$$\begin{cases} \ddot{x}_1 - 2\left(\frac{\bar{\theta}}{4} + \theta\omega^2\right)\dot{x}_2 - \left(1 - \frac{\theta\bar{\theta}}{4}\right)^2 \omega^2 x_1 = 0, \\ \ddot{x}_2 - 2\left(\frac{\bar{\theta}}{4} + \theta\omega^2\right)\dot{x}_1 - \left(1 - \frac{\theta\bar{\theta}}{4}\right)^2 \omega^2 x_2 = 0. \end{cases} \tag{24}$$

Note that upon setting $\theta = \bar{\theta} = 0$, we get back (9). Solutions of (24) can be written as follows

$$\begin{cases} x_1(t) = K_1 e^{(\eta+\xi)t} + K_2 e^{(\eta-\xi)t} + K_3 e^{-(\eta-\xi)t} + K_4 e^{-(\eta+\xi)t}, \\ x_2(t) = K_1 e^{(\eta+\xi)t} + K_2 e^{(\eta-\xi)t} - K_3 e^{-(\eta-\xi)t} - K_4 e^{-(\eta+\xi)t}, \end{cases} \tag{25}$$

where

$$\eta = \theta\omega^2 + \frac{\bar{\theta}}{4}, \qquad \xi = (\eta^2 + \lambda)^{1/2} \quad \text{and} \quad \lambda = \omega^2 \left(1 - \frac{\theta\bar{\theta}}{4}\right)^2, \tag{26}$$

with K_1, K_2, K_3 and K_4 being the constants of integration. The Hamiltonian constraint, $\mathcal{H}' = 0$, leads to

$$K_3 + K_4 = 0 \quad \text{or} \quad K_1 = K_2 = 0, \tag{27}$$

Thus, from (4) and (25) we can recover the scale factor and scalar field $\phi(t)$ as

$$\begin{cases} R(t)^3 = e^{-2\xi t}[K_1 e^{2\xi t} + K_2][K_3 e^{2\xi t} + K_4], \\ \phi(t) = \frac{1}{\alpha}\tanh^{-1}\left(\frac{K_1 e^{(\eta+\xi)t} + K_2 e^{(\eta-\xi)t} - K_3 e^{-(\eta-\xi)t} - K_4 e^{-(\eta+\xi)t}}{K_1 e^{(\eta+\xi)t} + K_2 e^{(\eta-\xi)t} + K_3 e^{-(\eta-\xi)t} + K_4 e^{-(\eta+\xi)t}}\right). \end{cases} \tag{28}$$

It is clear from the solutions (28), that we still have a late time de Sitter phase solution

$$R^3 = K_1 K_3 e^{2\xi t}, \tag{29}$$

and an initial Big Bang singularity. However, (29) dictates the new cosmological constant in the noncommutative case as

$$\Lambda_{nc} = \frac{4}{3}\left(1 + \theta^2\omega^2\right)\left(\omega^2 + \frac{1}{16}\bar{\theta}^2\right) = \left(1 + \frac{3}{4}\theta^2\Lambda\right)\left(\Lambda + \frac{1}{12}\bar{\theta}^2\right). \tag{30}$$

The effective cosmological constant on the other hand, is a measure of the ultraviolet cutoff in the theory. Hence, the above equation can be interpreted as a redefinition of the cutoff in the theory due the presence of noncommutativity. In this regard, according to [35] we take

$$\Lambda \sim M_{EW}^4, \qquad \theta^2 \sim M_{EW}^{-4}, \quad \text{and} \quad \bar{\theta}^2 \sim M_P^4, \tag{31}$$

where M_{EW} is the electroweak mass scale. Therefore, from the above relations we obtain

$$\Lambda_{nc} = M_P^4, \tag{32}$$

which defines the cutoff in the noncommutative model. In other words if we assume M_{EW} to be the natural cutoff in the original commutative model, the Planck mass is a cutoff in the noncommutative model. A similar discussion can be found for a classical cosmological model in [19] and at the quantum level in [35].

4 Generalized Uncertainty Principle (GUP) Approach

According to [36], the modified commutation relations in GUP approach for more than one dimensional systems is given by

$$\{x'_i, p'_j\} = \delta_{ij}(1 + \beta p'^2) + \beta' p'_i p'_j, \tag{33}$$

where β and β' are constant. If we assume the components of momenta, p'_i, to commute with each other i.e.

$$\{p'_i, p'_j\} = 0, \tag{34}$$

then the commutation relations among the coordinates x'_i are almost uniquely determined by the Jacobi identity as [37, 38]

$$\{x'_i, x'_j\} = \frac{(2\beta - \beta') + (2\beta + \beta')\beta p'^2}{1 + \beta p'^2}(p'_i x'_j - p'_j x'_i). \tag{35}$$

Similar to the previous section, we can introduce the following transformations

$$\begin{cases} x'_i = \left[(1 + \beta p^2)x_i + \beta' p_i p_j x_i + \gamma p_i\right], \\ p'_i = p_i. \end{cases} \tag{36}$$

One can show that if (x_i, p_j) obey the ordinary Poisson algebra (7), then (x'_i, p'_j) satisfy the deformed commutation relation (33) with the usual Poisson brackets. The arbitrary constant γ in the representation of x'_i does not affect the commutation relation among the x'_i's and is given by [39]

$$\gamma = \beta + \beta' \left(\frac{D+1}{2}\right). \tag{37}$$

Note that for $D = 2$, we have $\gamma = \beta + \frac{3}{2}\beta'$ and $p^2 = p_1^2 + p_1^2$. Moreover, β and $\beta' > 0$ and are assumed small up to the first order.

The minimal variation of Hamiltonian (6) in the GUP approach then becomes

$$\begin{aligned}\mathcal{H}' &= \frac{1}{4}(p_1'^2 - p_2'^2) - \omega^2(x_1'^2 - x_2'^2)\\ &= \frac{1}{4}(p_1^2 - p_2^2) - \omega^2\{(1+2\beta p^2)(x_1^2 - x_2^2) + 2(x_1p_1 - x_2p_2)[\gamma + \beta'(x_1p_1 + x_2p_2)]\}.\end{aligned} \tag{38}$$

The equations of motion corresponding to the above modified Hamiltonian (38) are given by

$$\begin{cases}\dot{x}_1 = \frac{p_1}{2} - \omega^2[4\beta(x_1^2 - x_2^2)p_1 + 2\beta'(x_1p_1 - x_2p_2)x_1 + 2(\gamma + \beta'(x_1p_1 + x_2p_2))x_1],\\ \dot{x}_2 = \frac{-p_2}{2} - \omega^2[4\beta(x_1^2 - x_2^2)p_2 + 2\beta'(x_1p_1 - x_2p_2)x_2 - 2(\gamma + \beta'(x_1p_1 + x_2p_2))x_2],\\ \dot{p}_1 = \omega^2[2(1+2\beta p^2)x_1 + 2\beta'(x_1p_1 - x_2p_2)p_1 + 2(\gamma + \beta'(x_1p_1 + x_2p_2))p_1],\\ \dot{p}_2 = \omega^2[-2(1+2\beta p^2)x_2 + 2\beta'(x_1p_1 - x_2p_2)p_2 - 2(\gamma + \beta'(x_1p_1 + x_2p_2))p_2].\end{cases} \tag{39}$$

After a little calculation, we obtain

$$\begin{cases}\ddot{x}_1 = \omega^2 x_1 - 8\omega^4(\beta+\beta')x_1^3 + 8\omega^4\beta x_1x_2^2 - 8\omega^2(\beta+\beta')x_1\dot{x}_1^2\\ \qquad + 16\omega^2\beta x_2\dot{x}_1\dot{x}_2 + 8\omega^2\beta x_1\dot{x}_2^2,\\ \ddot{x}_2 = \omega^2 x_2 - 8\omega^4\beta' x_2^3 - 8\omega^2\beta(x_1^2 - x_2^2)\dot{x}_1 + 8\omega^2\beta x_2\dot{x}_1^2\\ \qquad + 16\omega^2\beta x_1\dot{x}_1\dot{x}_2 - 8\omega^2(\beta+\beta')x_2\dot{x}_2^2.\end{cases} \tag{40}$$

At this stage, we solve (40) using the perturbation method to expand new solutions around commutative solutions as

$$x_i = x_{0i} + \beta f_i(t) + \beta' g_i(t), \tag{41}$$

where x_{0i} obey the commutative solutions. After substituting the expansion (41) into (40) and ignoring higher orders of β, β' and ω^2 terms we obtain

$$\begin{cases}\ddot{f}_i = \omega^2 f_i,\\ \ddot{g}_i = \omega^2 g_i.\end{cases} \tag{42}$$

Now inserting the solutions of above equations into (41) and using the commutative solutions (10), we obtain

$$\begin{cases}x_1 = (A_1 + \beta M_1 + \beta' M_3)e^{\omega t} + (A_2 + \beta M_2 + \beta' M_4)e^{-\omega t},\\ x_2 = (A_2 + \beta M_5 + \beta' M_7)e^{\omega t} + (A_1 + \beta M_6 + \beta' M_8)e^{-\omega t}.\end{cases} \tag{43}$$

On the other hand the Hamiltonian constraint (38) gives the following relations between the integration constants

$$\begin{cases}A_1(M_2 - M_5) + A_2(M_1 - M_6) = 0,\\ A_1(M_4 - M_7) + A_2(M_3 - M_8) = 0.\end{cases} \tag{44}$$

Finally, using (4) the scale factor and scalar field become

$$R(t)^3 = (A_1^2 - A_2^2)\sinh(2\omega t) + 2[\beta(A_1M_1 - A_2M_5) + \beta'(A_1M_3 - A_2M_7)]e^{2\omega t} + 2[\beta(A_2M_2 - A_1M_6) + \beta'(A_2M_4 - A_1M_8)]e^{-2\omega t}, \tag{45}$$

and

$$\phi(t) = \frac{1}{\alpha}\tanh^{-1}\left[\frac{(A_2 + \beta M_5 + \beta' M_7)e^{\omega t} + (A_1 + \beta M_6 + \beta' M_8)e^{-\omega t}}{(A_1 + \beta M_1 + \beta' M_3)e^{\omega t} + (A_2 + \beta M_2 + \beta' M_4)e^{-\omega t}}\right]. \tag{46}$$

Solution (45) shows that we have a late de Sitter phase, like commutative and Moyal noncommutative cases, with unchanged cosmological constant. Hence in the GUP modification of this simple cosmological model, the cosmological constant remains unchanged. However, solution (45), dose not have a curvature singularity and it seems the GUP approach addresses the initial Big Bang singularity problem.

5 Conclusions

In this manuscript we have introduced two different deformation methods between scale factor of the FRW universe and the scalar field in minisuperspace. We have shown that the classical solutions of such models clearly point to a possible resolution of the cosmological constant problem using the Moyal product deformation and the initial Big Bang singularity wither GUP approach. It appears that noncommutative models in minisuperspace in conjunction with of the both Moyal and GUP methods [40] can clarify simultaneously the two above mentioned problems.

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